5. **Solve the following boundary value problem**

$$\frac{d^2y}{dt^2} + 9y = \cos t \text{ where } y(0) = 1 \text{ and } y\left(\frac{\pi}{2}\right) = 2.$$ [CP 2016 Type]

Solution Here the given differential equation is

$$\frac{d^2y}{dt^2} + 9y = \cos t \qquad \dots (1)$$

Taking Laplace transform of both sides of (1) we have,

$$L\left\{\frac{d^2y}{dt^2} + 9y\right\} = L\{\cos t\}$$

$$\text{or, } L\left\{\frac{d^2y}{dt^2}\right\} + 9L\{y\} = \frac{p}{1+p^2}$$

$$\text{or, } p^2L\{y\} - py(0) - \left.\frac{dy}{dt}\right|_{t=0} + 9\bar{y} = \frac{p}{1+p^2} \quad [\text{where } \bar{y} = L\{y\}]$$

$$\text{or, } (p^2+9)\bar{y} - p\cdot 1 - c = \frac{p}{1+p^2} \text{ since } y(0) = 1 \text{ and } \left.\frac{dy}{dt}\right|_{t=0} = c, \text{ say}$$

$$\text{or, } \bar{y} = \frac{c+p}{9+p^2} + \frac{p}{(1+p^2)(9+p^2)} = \frac{p}{9+p^2} + \frac{c}{3}\cdot\frac{3}{9+p^2} + \frac{1}{8}\left[\frac{p}{1+p^2} - \frac{p}{9+p^2}\right]$$

Taking inverse Laplace transform of both sides, we have,

$$y(t) = L^{-1}\left\{\frac{p}{9+p^2}\right\} + \frac{c}{3}L^{-1}\left\{\frac{3}{9+p^2}\right\} + \frac{1}{8}\left[L^{-1}\left\{\frac{p}{1+p^2}\right\} - L^{1}\left\{\frac{p}{9+p^2}\right\}\right]$$

$$= \cos 3t + \frac{c}{3}\sin 3t + \frac{1}{8}(\cos t - \cos 3t)$$

$$= \frac{1}{8}\cos t + \frac{7}{8}\cos 3t + \frac{c}{3}\sin 3t \qquad \dots (2)$$

To determine c, substituting $t = \frac{\pi}{2}$ and noting $y\left(\frac{\pi}{2}\right) = 2$, we have,

$$y\left(\frac{\pi}{2}\right) = 2 = \frac{1}{8}\cos\left(\frac{\pi}{2}\right) + \frac{7}{8}\cos\left(\frac{3\pi}{2}\right) + \frac{c}{3}\sin\left(\frac{3\pi}{2}\right)$$

or, $-\frac{c}{3} = 2$ giving $c = -6$.

Hence putting $c = -6$ in (2), the required solution becomes :

$$y(t) = \frac{1}{8}\cos t + \frac{7}{8}\cos 3t - 2\sin 3t.$$

6. **Solve the equation** $\frac{d^2x}{dt^2} + \lambda^2 x = a\cos\omega t$; **given that** $x = c_0$ **and** $\frac{dx}{dt} = c_1$ **when** $t = 0$.

Solution Here the given differential equation is

$$\frac{d^2x}{dt^2} + \lambda^2 x = a\cos\omega t \qquad \dots (1)$$

Taking Laplace transform of the both sides of (1) we have,

$$L\left\{\frac{d^2x}{dt^2} + \lambda^2 x\right\} = L\{a\cos\omega t\} \text{ or, } L\left\{\frac{d^2x}{dt^2}\right\} + \lambda^2 L\{x\} = \frac{ap}{\omega^2 + p^2}$$

or, $p^2 L\{x\} - px(0) - \left.\frac{dx}{dt}\right|_{t=0} + \lambda^2 \bar{x} = \frac{ap}{\omega^2 + p^2}$ [where $\bar{x} = L\{x(t)\}$]

or, $p^2 \bar{x} + \lambda^2 \bar{x} - p \cdot c_0 - c_1 = \frac{ap}{\omega^2 + p^2}$ [using the initial conditions]

or, $(\lambda^2 + p^2)\bar{x} = \frac{ap}{\omega^2 + p^2} + (pc_0 + c_1)$

or, $\bar{x} = \frac{ap}{(\lambda^2 + p^2)(\omega^2 + p^2)} + \frac{pc_0 + c_1}{\lambda^2 + p^2} = c_0 \frac{p}{\lambda^2 + p^2} + \frac{c_1}{\lambda} \cdot \frac{\lambda}{\lambda^2 + p^2} + \frac{a}{\lambda^2 - \omega^2}\left(\frac{p}{\omega^2 + p^2} - \frac{p}{\lambda^2 + p^2}\right)$

Taking inverse Laplace transform from both sides, we have,

$$x(t) = c_0 L^{-1}\left\{\frac{p}{\lambda^2 + p^2}\right\} + \frac{c_1}{\lambda} L^{-1}\left\{\frac{\lambda}{\lambda^2 + p^2}\right\} + \frac{a}{\lambda^2 - \omega^2} L^{-1}\left\{\frac{p}{\omega^2 + p^2} - \frac{p}{\lambda^2 + p^2}\right\}$$

$$= c_0 \cos\lambda t + \frac{c_1}{\lambda}\sin\lambda t + \frac{a}{\lambda^2 - \omega^2}(\cos\omega t - \cos\lambda t)$$

which is the required solution of the given differential equation.

7. **Solve using Laplace transform $(D^2 + 4)y = 4$ given that $y = 1$ and $Dy = 5$ when $t = 0$ $\left(D \equiv \frac{d}{dt}\right)$.** [CP 2004, 2013]

Solution Here the given differential equation is :

$(D^2 + 4)y = 4$ ... (1)

Taking Laplace transform of (1) we have,

$L\{(D^2 + 4)y\} = L\{4\}$

or, $L\{D^2 y\} + 4L\{y\} = \frac{4}{p}$

or, $p^2 L\{y\} - py(0) - \left.\frac{dy}{dt}\right|_{t=0} + 4L\{y\} = \frac{4}{p}$

or, $(p^2 + 4)L\{y\} - p \cdot 1 - 5 = \frac{4}{p}$ [using initial conditions]

or, $(p^2 + 4)L\{y\} = \frac{4}{p} + p + 5$

or, $L\{y\} = \frac{4}{p(p^2 + 4)} + \frac{p + 5}{p^2 + 4}$

Taking inverse Laplace transform, we have

$$y(t) = L^{-1}\left\{\frac{4}{p(p^2 + 4)} - \frac{p + 5}{p^2 + 4}\right\} = L^{-1}\left\{\frac{1}{p} - \frac{p}{p^2 + 4} + \frac{p + 5}{p^2 + 4}\right\}$$

$$= L^{-1}\left\{\frac{1}{p}\right\} + \frac{5}{2} L^{-1}\left\{\frac{2}{p^2 + 4}\right\} = 1 + \frac{5}{2}\sin 2t$$

which is the required solution.

8. **Solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$, given $y(0) = 0 = y'(0)$ using Laplace transformation method.** [CP 2017]

Solution Here the given differential equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t} \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1) we have,

$$L\left\{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y\right\} = L\{e^{-t}\}$$

or, $L\left\{\frac{d^2y}{dt^2}\right\} + 3L\left\{\frac{dy}{dt}\right\} + 2L\{y\} = \frac{1}{p+1}$

or, $p^2L\{y\} - py(0) - y'(0) + 3[pL\{y\} - y(0)] + 2L\{y\} = \frac{1}{p+1}$

or, $(p^2 + 3p + 2)L\{y\} - p\cdot 0 - 0 = \frac{1}{p+1}$ [using initial conditions]

or, $L\{y\} = \frac{1}{(p+1)(p^2+3p+2)} = \frac{1}{(p+1)^2(p+2)}$

Taking inverse Laplace transform, we have,

$$y(t) = L^{-1}\left\{\frac{1}{(p+1)^2(p+2)}\right\} = L^{-1}\left\{\frac{1}{(p+1)^2} - \frac{1}{(p+1)(p+2)}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p+1)^2} - \frac{1}{p+1} + \frac{1}{p+2}\right\} = L^{-1}\left\{\frac{1}{(p+1)^2}\right\} - L^{-1}\left\{\frac{1}{p+1}\right\} + L^{-1}\left\{\frac{1}{p+2}\right\}$$

$$= te^{-t} - e^{-t} + e^{-2t} = (t-1)e^{-t} + e^{-2t}$$

Hence the required solution is :

$$y(t) = (t-1)e^{-t} + e^{-2t}.$$

9. **Solve by Laplace transformation :**

$(D^2 + D)x = 2$, $x(0) = 3$, $x'(0) = 1$, $D \equiv \frac{d}{dt}$. [CP 2018]

Solution Here the given differential equation is :

$$(D^2 + D)x = 2 \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\{(D^2 + D)x\} = L\{2\}$$

or, $L\{D^2x\} + L\{Dx\} = \frac{2}{p}$

or, $p^2L\{x\} - px(0) - x'(0) + pL\{x\} - x(0) = \frac{2}{p}$

or, $(p^2 + p)L\{x\} - p\cdot 3 - 1 - 3 = \frac{2}{p}$ [Using initial conditions] or, $(p^2 + p)L\{x\} = \frac{2}{p} + 3p + 4 = \frac{3p^2+4p+2}{p}$

or, $L\{x\} = \frac{3p^2+4p+2}{p^2(p+1)} = \frac{3}{p+1} + \frac{4}{p(p+1)} + \frac{2}{p^2(p+1)}$

Taking inverse Laplace transform of both sides, we have,

$$x(t) = L^{-1}\left\{\frac{3p^2+4p+2}{p^2(p+1)}\right\} = L^{-1}\left\{\frac{3}{p+1}+\frac{4}{p(p+1)}+\frac{2}{p^2(p+1)}\right\}$$

$$= L^{-1}\left\{\frac{3}{p+1}+\frac{4}{p}-\frac{4}{p+1}+\frac{2}{p^2}-\frac{2}{p(p+1)}\right\}$$

$$= L^{-1}\left\{\frac{2}{p^2}+\frac{4}{p}-\frac{1}{p+1}-\frac{2}{p}+\frac{2}{p+1}\right\} = L^{-1}\left\{\frac{2}{p^2}+\frac{2}{p}+\frac{1}{p+1}\right\}$$

$$= L^{-1}\left\{\frac{2}{p^2}\right\} + L^{-1}\left\{\frac{2}{p}\right\} + L^{-1}\left\{\frac{1}{p+1}\right\}$$

$$= 2t + 2 + e^{-t}$$

Hence the required solution is : $x(t) = 2t + 2 + e^{-t}$.

10. **Solve the initial value problem : $(D^2 - 3D + 2)y = 1 - e^{2t}$, where $y = 1$ and $Dy = 0$ when $t = 0$ and $D \equiv \frac{d}{dt}$.**

Solution Here the given differential equation is

$$(D^2 - 3D + 2)y = 1 - e^{2t},\ D \equiv \frac{d}{dt} \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$L\{(D^2 - 3D + 2)y\} = L\{1 - e^{2t}\}$

or, $L\{D^2y\} - 3L\{Dy\} + 2L\{y\} = L\{1\} - L\{e^{2t}\}$

or, $p^2L\{y\} - py(0) - Dy(0) - 3[pL\{y\} - y(0)] + 2L\{y\} = \frac{1}{p} - \frac{1}{p-2}$

or, $(p^2 - 3p + 2)\bar{y} - p\cdot 1 - 0 + 3\cdot 1 = -\frac{2}{p(p-2)}$ [using initial conditions and $\bar{y} = L\{y\}$]

or, $(p-1)(p-2)\bar{y} = p - 3 - \frac{2}{p(p-2)} = \frac{p^3-5p^2+6p-2}{p(p-2)} = \frac{(p-1)(p^2-4p+2)}{p(p-2)}$

or, $\bar{y} = \frac{p^2-4p+2}{p(p-2)^2} = \frac{1}{2p} + \frac{1}{2(p-2)} - \frac{1}{(p-2)^2}$

Taking inverse Laplace transform from the both sides we have,

$$y = L^{-1}\left\{\frac{1}{2p}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p-2}\right\} - L^{-1}\left\{\frac{1}{(p-2)^2}\right\}$$

i.e., $y(t) = \frac{1}{2} + \frac{1}{2}e^{2t} - te^{2t}$, which is the required solution of the given differential equation.

11. Using Laplace transform, solve the equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4e^{2x}$ when $y(0) = -3$ and $\left(\frac{dy}{dx}\right)_{x=0} = 5$. [CP 2014, 2013, 2009]

Solution Here the given differential equation is :

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4e^{2x} \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\left\{\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y\right\} = L\{4e^{2x}\} \quad \text{or, } L\left\{\frac{d^2y}{dx^2}\right\} - 3L\left\{\frac{dy}{dx}\right\} + 2L\{y\} = \frac{4}{p-2}$$

$$\text{or, } \left[p^2L\{y\} - py(0) - \left(\frac{dy}{dx}\right)_{x=0}\right] - 3[pL\{y\} - y(0)] + 2L\{y\} = \frac{4}{p-2}$$

$$\text{or, } (p^2 - 3p + 2)L\{y\} - (p-3)(-3) - 5 = \frac{4}{p-2} \quad \text{[using initial conditions]}$$

$$\text{or, } (p-1)(p-2)L\{y\} = \frac{4}{p-2} + 5 - 3(p-3) = \frac{4}{p-2} - 3p + 14$$

$$\text{or, } L\{y\} = \frac{4}{(p-1)(p-2)^2} - \frac{3p}{(p-1)(p-2)} + \frac{14}{(p-1)(p-2)}$$

$$= \frac{4}{(p-2)^2} - \frac{4}{(p-1)(p-2)} + \frac{14}{(p-1)(p-2)} - \frac{3}{2}\frac{(p-1)+(p-2)+3}{(p-1)(p-2)}$$

$$= \frac{4}{(p-2)^2} - \frac{10}{(p-1)(p-2)} - \frac{3}{2}\cdot\frac{1}{p-2} - \frac{3}{2}\cdot\frac{1}{p-1} - \frac{9}{2}\cdot\frac{1}{(p-1)(p-2)}$$

$$= \frac{4}{(p-2)^2} - \frac{3}{2}\left(\frac{1}{p-1} + \frac{1}{p-2}\right) - \frac{29}{2}\left(\frac{1}{p-2} - \frac{1}{p-1}\right)$$

$$= \frac{4}{(p-2)^2} - \frac{16}{p-2} + \frac{13}{p-1}$$

Taking inverse Laplace transform, we have

$$y(t) = L^{-1}\left\{\frac{4}{(p-2)^2} - \frac{16}{p-2} + \frac{13}{p-1}\right\}$$

$$= 4te^{2t} - 16e^{2t} + 13e^{t} = 4e^{2t}(t-4) + 13e^{t}$$

which is the required solution.

12. Solve using Laplace transformation, the equation $\frac{d^2y}{dt^2} + y = \sin 3t$ when $y(0) = 0$, $\left(\frac{dy}{dt}\right)_{t=0} = 0$. [CP 2014]

Solution Here the given differential equation is

$$\frac{d^2y}{dt^2} + y = \sin 3t \quad \text{... (1)}$$

Taking Laplace transform both sides of (1) we have,

$$L\left\{\frac{d^2y}{dt^2}+y\right\} = L\{\sin 3t\}$$

$$\text{or, } L\left\{\frac{d^2y}{dt^2}\right\} + L\{y\} = \frac{3}{p^2+9}$$

$$\text{or, } p^2L\{y\} - py(0) - \left(\frac{dy}{dt}\right)_{t=0} + L\{y\} = \frac{3}{p^2+9}$$

$$\text{or, } (p^2+1)L\{y\} = \frac{3}{p^2+9} \quad \left[\text{since } y(0) = \left(\frac{dy}{dt}\right)_{t=0} = 0, \text{ given}\right]$$

$$\text{or, } L\{y\} = \frac{3}{(p^2+1)(p^2+9)}$$

Taking inverse Laplace transform of both sides, we have,

$$y(t) = L^{-1}\left\{\frac{3}{(p^2+1)(p^2+9)}\right\} = L^{-1}\left\{\frac{3}{8}\left(\frac{1}{p^2+1}-\frac{1}{p^2+9}\right)\right\} = \frac{3}{8}L^{-1}\left\{\frac{1}{p^2+1}\right\} - \frac{1}{8}L^{-1}\left\{\frac{3}{p^2+9}\right\}$$

$$= \frac{3}{8}\sin t - \frac{1}{8}\sin 3t = \frac{1}{8}(3\sin t - \sin 3t),$$

which is the required solution.

13. **Solve the initial value problem using Laplace transform :**

$$\frac{d^2x}{dt^2} + 4x = \sin 3t,\ x(0) = 0 \text{ and } x'(0) = 0.$$

Solution Here the given differential equation is

$$\frac{d^2x}{dt^2} + 4x = \sin 3t \qquad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\left\{\frac{d^2x}{dt^2}\right\} + 4L\{x\} = L\{\sin 3t\}$$

$$\text{or, } p^2L\{x\} - px(0) - x'(0) + 4L\{x\} = \frac{3}{p^2+(3)^2}$$

$$\text{or, } (p^2+4)\bar{x} - p\cdot 0 - 0 = \frac{3}{p^2+9} \quad [\text{where } \bar{x} = L\{x\} \text{ and since } x(0) = x'(0) = 0]$$

$$\text{or, } \bar{x} = \frac{3}{(p^2+4)(p^2+9)} = \frac{3}{5}\left[\frac{1}{p^2+4}-\frac{1}{p^2+9}\right]$$

Taking inverse Laplace transform of both sides, we have,

$$x(t) = \frac{3}{5}\left[L^{-1}\left\{\frac{1}{p^2+4}\right\} - L^{-1}\left\{\frac{1}{p^2+9}\right\}\right] = \frac{3}{5}\left[\frac{1}{2}\sin 2t - \frac{1}{3}\sin 3t\right]$$

$$= \frac{1}{10}[3\sin 2t - 2\sin 3t]$$

which is the required solution of the given differential equation.

14. **Solve the boundary value problem :**

$(D+1)^2 y = t$, given that $y = -3$, when $t = 0$ and $y = -1$ when $t = 1$ and $D \equiv \frac{d}{dt}$.

[CP 2020]

Solution Here the given differential equation is :

$$(D+1)^2 y = t \text{ i.e., } (D^2 + 2D + 1)y = t \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\{D^2y\} + 2L\{Dy\} + L\{y\} = L\{t\}$$

$$\text{or, } [p^2L\{y\} - py(0) - y'(0)] + 2[pL\{y\} - y(0)] + L\{y\} = \frac{1}{p^2}$$

$$\text{or, } (p^2 + 2p + 1)\bar{y} - p\cdot(-3) - C - 2\cdot(-3) = \frac{1}{p^2}$$

where $\bar{y} = L\{y\}$, $y'(0) = c$, say and $y(0) = -3$, given

$$\text{or, } (p+1)^2\bar{y} = \frac{1}{p^2} + c - 6 - 3p.$$

$$\text{or, } \bar{y} = \frac{1}{p^2(p+1)^2} + \frac{c}{(p+1)^2} - \frac{3p+6}{(p+1)^2} \quad \text{... (2)}$$

Taking inverse Laplace transform of both sides of (2), we have,

$$y = L^{-1}\left\{\frac{1}{p^2(p+1)^2}\right\} + L^{-1}\left\{\frac{c}{(p+1)^2}\right\} - L^{-1}\left\{\frac{3p+6}{(p+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p+1)^2} + \frac{1}{p^2} + \frac{2}{p+1} - \frac{2}{p}\right\} + cL^{-1}\left\{\frac{1}{(p+1)^2}\right\} - L^{-1}\left\{\frac{3}{p+1}\right\} - L^{-1}\left\{\frac{3}{(p+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{p^2}\right\} - L^{-1}\left\{\frac{2}{p}\right\} - L^{-1}\left\{\frac{1}{p+1}\right\} + (c-2)L^{-1}\left\{\frac{1}{(p+1)^2}\right\}$$

$$= t - 2 - e^{-t}L^{-1}\left\{\frac{1}{p}\right\} + (c-2)e^{-t}\left\{\frac{1}{p^2}\right\}$$

$$= t - 2 - e^{-t}\cdot 1 + (c-2)\cdot e^{-t}\cdot t$$

$$= t - e^{-t} + (c-2)te^{-t} - 2 \quad \text{... (3)}$$

Using the boundary condition $y = -1$ when $t = 1$, we have from (3),

$$-1 = 1 - e^{-1} + (c-2)e^{-1} - 2$$

or, $c - 2 = 1$ giving $c = 3$.

substituting $c = 3$ in (3); we have the required solution as

$$y(t) = t - e^{-t} + (3-2)te^{-t} - 2 \text{ i.e., } y(t) = e^{-t}(t-1) + t - 2$$

15. **Solve by using Laplace transformation, the equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, given that when $t = 0$, $y = 0$ and $\frac{dy}{dt} = 0$.**

[CP 2016, 2008]

Solution Here the given differential equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\left\{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y\right\} = L\{e^{-t}\sin t\} \text{ or, } L\left\{\frac{d^2y}{dt^2}\right\} + 2\left\{\frac{dy}{dt}\right\} + 5L\{y\} = \frac{1}{(p+1)^2+1}$$

or, $\left[p^2L\{y\} - py(0) - \left.\frac{dy}{dt}\right|_{t=0}\right] + 2[pL\{y\} - y(0)] + 5L\{y\} = \frac{1}{p^2+2p+2}$

or, $(p^2 + 2p + 5)L\{y\} = \frac{1}{p^2+2p+2}$ [using initial conditions]

or, $L\{y\} = \frac{1}{(p^2+2p+2)(p^2+2p+5)}$

Taking inverse Laplace transform, we have,

$$y(t) = L^{-1}\left\{\frac{1}{(p^2+2p+2)(p^2+2p+5)}\right\} = \frac{1}{3}L^{-1}\left\{\frac{1}{(p+1)^2+1} - \frac{1}{(p+1)^2+4}\right\}$$

$$= \frac{e^{-t}}{3}\left[L^{-1}\left\{\frac{1}{p^2+1}\right\} - L^{-1}\left\{\frac{1}{p^2+4}\right\}\right] \text{ [using first shifting property]}$$

$$= \frac{e^{-t}}{3}\left[\sin t - \frac{\sin 2t}{2}\right] = \frac{e^{-t}}{6}[2\sin t - \sin 2t]$$

which is the required solution.

16. **Solve the initial value problem using Laplace transform :**

$(D^2 + 2D + 5)y = e^{-t}\sin t, y(0) = 0, y'(0) = 1$ and $D \equiv \frac{d}{dt}$.

Solution Here the given differential equation is

$(D^2 + 2D + 5)y = e^{-t}\sin t$ ... (1)

Taking Laplace transform of both sides of (1), we have,

$L\{D^2y\} + 2L\{Dy\} + 5L\{y\} = L\{e^{-t}\sin t\}$

or, $[p^2L\{y\} - py(0) - y'(0)] + 2[pL\{y\} - y(0)] + 5L\{y\} = \frac{1}{1+(p+1)^2}$ [using first shifting property]

or, $(p^2 + 2p + 5)\bar{y} - p\cdot 0 - 1 - 2\cdot 0 = \frac{1}{p^2+2p+2}$ [using initial conditions and $\bar{y} = L\{y\}$]

or, $(p^2 + 2p + 5)\bar{y} = 1 + \frac{1}{p^2+2p+2} = \frac{p^2+2p+3}{p^2+2p+2}$

or, $\bar{y} = \frac{p^2+2p+3}{(p^2+2p+2)(p^2+2p+5)} = \frac{(p+1)^2+2}{\{(p+1)^2+1\}\{(p+1)^2+4\}}$

Taking inverse Laplace transform of both sides, we have,

$$y(t) = L^{-1}\left\{\frac{(p+1)^2+2}{[(p+1)^2+1][(p+1)^2+4]}\right\} = e^{-t}L^{-1}\left\{\frac{p^2+2}{(p^2+1)(p^2+4)}\right\} \text{ [using first shifting property]}$$

$$= e^{-t}L^{-1}\left\{\frac{1}{3}\left(\frac{1}{p^2+1} + \frac{2}{p^2+4}\right)\right\} = \frac{e^{-t}}{3}\left[L^{-1}\left\{\frac{1}{p^2+1}\right\} + L^{-1}\left\{\frac{2}{p^2+4}\right\}\right] = \frac{e^{-t}}{3}(\sin t + \sin 2t)$$

Hence the required solution of the given differential equation is $y(t) = \frac{e^{-t}}{3}(\sin t + \sin 2t)$

17. **Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 3te^{-t}$, given that $y(0) = 4$ and $y'(0) = 2$ by Laplace transformation.** [CP 2018]

Solution Here the given differential equation is :

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 3te^{-t} \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\left\{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y\right\} = L\{3te^{-t}\}$$

or, $L\left\{\frac{d^2y}{dt^2}\right\} + 2L\left\{\frac{dy}{dt}\right\} + L\{y\} = \frac{3}{(p+1)^2}$ [using first shifting property]

or, $p^2L\{y\} - py(0) - y'(0) + 2\{pL\{y\} - y(0)\} + L\{y\} = \frac{3}{(p+1)^2}$

or, $(p^2 + 2p + 1)\bar{y} - p\cdot 4 - 2 - 2\cdot 4 = \frac{3}{(p+1)^2}$ [using initial conditions and $\bar{y} = L\{y\}$]

or, $(p + 1)^2\bar{y} = \frac{3}{(p+1)^2} + 4p + 10$

or, $\bar{y} = \frac{3}{(p+1)^4} + \frac{4p}{(p+1)^2} + \frac{10}{(p+1)^2}$

Taking inverse Laplace transform both sides, we have,

$$y(t) = L^{-1}\left\{\frac{3}{(p+1)^4} + \frac{4p}{(p+1)^2} + \frac{10}{(p+1)^2}\right\} = L^{-1}\left\{\frac{3}{(p+1)^4} + \frac{4}{p+1} + \frac{6}{(p+1)^2}\right\}$$

$$= L^{-1}\left\{\frac{3}{(p+1)^4}\right\} + L^{-1}\left\{\frac{6}{(p+1)^2}\right\} = e$$

$$= e^{-t}L^{-1}\left\{\frac{3}{p^4} + \frac{6}{p^2} + \frac{4}{p}\right\} \text{ [using first shifting property]}$$

$$= e^{-t}\left[\frac{t^3}{2} + 6t + 4\right] = \frac{e^{-t}}{2}[t^3 + 12t + 8]$$

which is the required solution.

18. **Solve the following differential equation by Laplace transform :**

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = t\cos t$; given that $y(0) = 0 = y'(0)$.

Solution Here the given differential equation is

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = t\cos t \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\left\{\frac{d^2y}{dt^2}\right\} - 2L\left\{\frac{dy}{dt}\right\} - 3L\{y\} = L\{t\cos t\}$$

or, $p^2L\{y\} - py(0) - y'(0) - 2pL\{y\} + y(0) - 3L\{y\} = -\frac{d}{dp}[L\{\cos t\}]$

or, $(p^2 - 2p - 3)\bar{y} - p\cdot 0 - 0 + 0 = -\frac{d}{dp}\left\{\frac{p}{1+p^2}\right\}$ [using initial conditions and $\bar{y} = L\{y\}$]

or, $(p + 1)(p - 3)\bar{y} = \frac{p^2-1}{(1+p^2)^2}$

or, $\bar{y} = \frac{p-1}{(p-3)(p^2+1)^2} = \frac{1}{50}\cdot\frac{1}{p-3} - \frac{3}{50}\cdot\frac{1}{p^2+1} - \frac{1}{50}\cdot\frac{p}{p^2+1} + \frac{2}{5}\cdot\frac{1}{(p^2+1)^2} - \frac{1}{5}\cdot\frac{p}{(p^2+1)^2}$

Taking inverse Laplace transform from both sides, we have,

$$y(t) = \frac{1}{50}L^{-1}\left\{\frac{1}{p-3}\right\} - \frac{3}{50}L^{-1}\left\{\frac{1}{p^2+1}\right\} - \frac{1}{50}L^{-1}\left\{\frac{p}{p^2+1}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} - \frac{1}{5}L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} \quad \text{... (2)}$$

We know that, $L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t$ and $L^{-1}\left\{\frac{p}{p^2+1}\right\} = \cos t$.

Also $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t\sin t}{2}$

To find $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$ by convolution theorem, we have,

$$L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{1+p^2}\cdot\frac{1}{1+p^2}\right\} = \int_0^t \sin x\sin(t-x)dx \text{ [since } L^{-1}\left\{\frac{1}{1+p^2}\right\} = \sin t]$$

$$= \frac{1}{2}\int_0^t [\cos(2x - t) - \cos t]dx = \frac{1}{2}\left[\frac{1}{2}\sin(2x-t) - x\cos t\right]_{x=0}^{t}$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin t - t\cos t + \frac{1}{2}\sin t\right) = \frac{1}{2}(\sin t - t\cos t)$$

Using above results, we have from (2),

$$y(t) = \frac{e^{3t}}{50} - \frac{3\sin t}{50} - \frac{\cos t}{50} + \frac{2}{5}\cdot\frac{1}{2}(\sin t - t\cos t) - \frac{1}{5}\cdot\frac{1}{2}t\sin t$$

$$= \frac{e^{3t}}{50} - \frac{3\sin t}{50} - \frac{\cos t}{50} - \frac{1}{10}t\sin t + \frac{1}{5}(\sin t - t\cos t)$$

which is the required solution of the given differential equation.

19. **Solve the initial value problem using Laplace transform :**

$(D^2 - 1)y = a\cosh nt$; given that $y(0) = 0$, $y'(0) = 2$ where $D \equiv \frac{d}{dt}$.

Solution Here the given differential equation is

$$(D^2 - 1)y = a\cosh nt \quad \text{... (1)}$$

Taking Laplace transform of both sides we have,

$L\{(D^2 - 1)y\} = L\{a\cosh nt\}$ or, $L\{D^2y\} - L\{y\} = aL\{\cosh nt\}$

or, $p^2L\{y\} - py(0) - y'(0) - L\{y\} = \frac{ap}{p^2 - n^2}$

or, $(p^2-1)\bar{y} - p\cdot 0 - 2 = \dfrac{ap}{p^2-n^2}$ [using initial conditions and $\bar{y} = L\{y\}$]

or, $\bar{y} = \dfrac{2}{p^2-1} + \dfrac{ap}{(p^2-n^2)(p^2-1)} = \dfrac{2}{p^2-1} + \dfrac{a}{n^2-1}\left\{\dfrac{p}{p^2-n^2} - \dfrac{p}{p^2-1}\right\}$

Taking inverse Laplace transform from both sides we have,

$$y(t) = L^{-1}\left\{\frac{2}{p^2-1}\right\} + \frac{a}{n^2-1}\left[L^{-1}\left\{\frac{p}{p^2-n^2}\right\} - L^{-1}\left\{\frac{p}{p^2-1}\right\}\right]$$

$$= 2\sinh t + \frac{a}{n^2-1}(\cosh nt - \cosh t)$$

which is the required solution of the given equation.

20. **Using Laplace transformation, solve the following initial value problem $\dfrac{d^2y}{dt^2} + 4y = 4H(t-2)$ with $y(0) = y'(0) = 0$ where $H(t-2)$ is the Heaviside unit step function.**

Solution Here the given differential equation is

$$\frac{d^2y}{dt^2} + 4y = 4H(t-2) \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have,

$$L\left\{\frac{d^2y}{dt^2} + 4y\right\} = L\{4H(t-2)\} \text{ or, } L\left\{\frac{d^2y}{dt^2}\right\} + 4L\{y\} = 4L\{H(t-2)\}$$

or, $p^2\bar{y} - py(0) - y'(0) + 4\bar{y} = 4\dfrac{e^{-2p}}{p}$ [where $\bar{y} = L\{y\}$]

or, $(p^2+4)\bar{y} - p\cdot 0 - 0 = 4\dfrac{e^{-2p}}{p}$ [using the initial conditions]

or, $\bar{y} = 4\dfrac{e^{-2p}}{p(p^2+4)}$.

$$\text{Therefore, } y(t) = L^{-1}\left\{4\frac{e^{-2p}}{p(p^2+4)}\right\} \quad \text{... (2)}$$

$$\text{Now, } L^{-1}\left\{\frac{4}{p(p^2+4)}\right\} = 2L^{-1}\left\{\frac{2}{p(p^2+4)}\right\} = 2\int_0^t \sin 2t\,dt = 1 - \cos 2t.$$

$$\text{Hence } L^{-1}\left\{\frac{4e^{-2p}}{p(p^2+4)}\right\} = [1 - \cos 2(t-2)]H(t-2)$$

So from (2) we have,

$y(t) = [1 - \cos 2(t-2)]H(t-2)$ which is the required solution.

21. **Solve the following initial value problem :**

$\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = 4t + e^{3t}$; given that $y(0) = 1$ and $y'(0) = -1$.

Solution Here the given differential equation is

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4t + e^{3t} \qquad \text{... (1)}$$

Taking Laplace transform of both sides of (1) we have,

$$L\left\{\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y\right\} = L\{4t + e^{3t}\}$$

or, $L\left\{\frac{d^2y}{dt^2}\right\} - 3L\left\{\frac{dy}{dt}\right\} + 2L\{y\} = 4L\{t\} + L\{e^{3t}\}$

or, $[p^2L\{y\} - py(0) - y'(0)] - 3[pL\{y\} - y(0)] + 2L(y) = \frac{4}{p^2} + \frac{1}{p-3}$

or, $(p^2 - 3p + 2)\bar{y} - p\cdot 1 + 1 + 3\cdot 1 = \frac{4p-12+p^2}{p^2(p-3)}$

[where $\bar{y} = L\{y\}$ and $y(0) = 1, y'(0) = -1$, given]

or, $(p-1)(p-2)\bar{y} = p - 4 + \frac{(p-2)(p+6)}{p^2(p-3)}$

or, $\bar{y} = \frac{p-4}{(p-1)(p-2)} + \frac{p+6}{p^2(p-3)(p-1)}$ [Assuming $p \neq 2$]

$$= \frac{3}{p-1} - \frac{2}{p-2} + \frac{p+6}{p^2(p-3)(p-1)} \qquad \text{... (2)}$$

Let $\frac{p+6}{p^2(p-3)(p-1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p-1} + \frac{D}{p-3}$,

where A, B, C, D are four constants to be determined.

or, $\frac{p+6}{p^2(p-3)(p-1)} = \frac{Ap(p-1)(p-3) + B(p-1)(p-3) + Cp^2(p-3) + Dp^2(p-1)}{p^2(p-3)(p-1)}$

or, $Ap(p-1)(p-3) + B(p-1)(p-3) + Cp^2(p-3) + Dp^2(p-1) = p + 6 \qquad \text{... (3)}$

Putting $p = 0$ in (3) we have,

$3B = 6$ giving $B = 2$

Putting $p = 1$ in (3) we have,

$C(1-3) = 7$ giving $C = -\frac{7}{2}$

Next, putting $p = 3$ in (3) we have,

$18D = 9$ giving $D = \frac{1}{2}$

Finally putting $p = 2$ in (3) we have,

$A\cdot 2\cdot 1\cdot(-1) + 2\cdot 1\cdot(-1) - \frac{7}{2}\cdot 4\cdot(-1) + \frac{1}{2}\cdot 4\cdot 1 = 8$

or, $-2A - 2 + 14 + 2 = 8$ giving $A = 3$.

Hence from (2) we have,

$$\bar{y} = \frac{3}{p-1} - \frac{2}{p-2} + \frac{3}{p} + \frac{2}{p^2} - \frac{7}{2}\cdot\frac{1}{p-1} + \frac{1}{2}\cdot\frac{1}{p-3}$$

Taking inverse Laplace transform of both sides we have,

$$y(t) = 3L^{-1}\left\{\frac{1}{p-1}\right\} - 2L^{-1}\left\{\frac{1}{p-2}\right\} + 3L^{-1}\left\{\frac{1}{p}\right\} + 2L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{7}{2}L^{-1}\left\{\frac{1}{p-1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p-3}\right\}$$

or, $y(t) = -\frac{1}{2}e^t - 2e^{2t} + 3 + 2t + \frac{1}{2}e^{3t} = \frac{1}{2}(e^{3t} - e^t) - 2e^{2t} + 2t + 3$

which is the required solution of the given differential equation.

22. **Solve the following system of simultaneous differential equations using Laplace transform :**

$$\frac{dx}{dt} - y = 0 \text{ and } \frac{dy}{dt} + x = 0 \text{ given that } x = 0 \text{ and } y = 1 \text{ at } t = 0.$$

Solution Here the given system of differential equation is :

$$\frac{dx}{dt} - y = 0 \quad \text{... (1)}$$

and $$\frac{dy}{dt} + x = 0 \quad \text{... (2)}$$

Taking the Laplace transform of (1) we have,

$$L\left\{\frac{dx}{dt}\right\} - L\{y\} = L\{0\}$$

or, $pL\{x\} - x(0) - L\{y\} = 0$

or, $p\bar{x} - 0 - \bar{y} = 0$ [where $\bar{x} = L\{x\}$, $\bar{y} = L\{y\}$ and $x(0) = 0$, given]

or, $\bar{y} = p\bar{x}$ ... (3)

Again taking the Laplace transform of (2) we have,

$$L\left\{\frac{dy}{dt}\right\} + L\{x\} = L\{0\}$$

or, $pL\{y\} - y(0) + L\{x\} = 0$

or, $p\bar{y} - 1 + \bar{x} = 0$ [since $y(0) = 1$, given]

or, $p^2\bar{x} - 1 + \bar{x} = 0$ [using (3)]

or, $\bar{x} = \frac{1}{1+p^2}$

Taking inverse Laplace transform, we have,

$$x(t) = L^{-1}\left\{\frac{1}{1+p^2}\right\} = \sin t$$

Again from (3) we have,

$$\bar{y} = p\bar{x} = \frac{p}{1+p^2}$$

Taking inverse Laplace transform, we have,

$$y(t) = L^{-1}\left\{\frac{p}{1+p^2}\right\} = \cos t$$

Hence the required solution of the given differential equation is : $x(t) = \sin t$, $y(t) = \cos t$.

23. **Solve the following system of simultaneous differential equations using Laplace transform :**

$\frac{dx}{dt} + y = \sin t$ **and** $\frac{dy}{dt} + x = \cos t$ **given that** $x(0) = 2$ **and** $y(0) = 0$.

Solution Here the given system of simultaneous differential equation is :

$$\frac{dx}{dt} + y = \sin t \quad \text{... (1)}$$

and $$\frac{dy}{dt} + x = \cos t \quad \text{... (2)}$$

Taking Laplace transform of (1) we have,

$$L\left\{\frac{dx}{dt}\right\} + L\{y\} = L\{\sin t\} \quad \text{or, } pL\{x\} - x(0) + L\{y\} = \frac{1}{1+p^2}$$

or, $p\bar{x} + \bar{y} - 2 = \frac{1}{1+p^2}$ [where $\bar{x} = L\{x\}$, $\bar{y} = L\{y\}$ and $x(0) = 2$, given]

or, $$\bar{y} = 2 - p\bar{x} + \frac{1}{1+p^2} \quad \text{... (3)}$$

Again taking Laplace transform of (2) we have,

$$L\left\{\frac{dy}{dt}\right\} + L\{x\} = L\{\cos t\}$$

or, $pL\{y\} - y(0) + L\{x\} = \frac{p}{1+p^2}$

or, $p\bar{y} - 0 + \bar{x} = \frac{p}{1+p^2}$ [since $y(0) = 0$, given]

or, $p\left[2 - p\bar{x} + \frac{1}{1+p^2}\right] + \bar{x} = \frac{p}{1+p^2}$ [using (3)]

or, $(1-p^2)\bar{x} + 2p + \frac{p}{1+p^2} = \frac{p}{1+p^2}$

or, $$\bar{x} = \frac{2p}{p^2-1} = \frac{1}{p-1} + \frac{1}{p+1} \quad \text{... (4)}$$

Taking inverse Laplace transform of (4) we have,

$$x(t) = L^{-1}\left\{\frac{1}{p-1}\right\} + L^{-1}\left\{\frac{1}{p+1}\right\} = e^t + e^{-t} \quad \text{... (5)}$$

Again using (4) in (3) we have,

$$\bar{y} = 2 - \frac{p}{p-1} - \frac{p}{p+1} + \frac{1}{1+p^2} = \frac{1}{1+p^2} + \frac{1}{p+1} - \frac{1}{p-1} \quad \text{... (6)}$$

Taking inverse Laplace transform of (6) we have,

$$y = L^{-1}\left\{\frac{1}{1+p^2}\right\} + L^{-1}\left\{\frac{1}{p+1}\right\} - L^{-1}\left\{\frac{1}{p-1}\right\} = \sin t + e^{-t} - e^t \quad \text{... (7)}$$

Combining (5) and (7), we have the required solution as follows :

$x(t) = e^t + e^{-t}$

$y(t) = \sin t + e^{-t} - e^t$

24. Find the general solution of $(D^2 + a^2)y = 0$ and verify it.

Solution Here the given differential equation is $(D^2 + a^2)y = 0$ ... (1)

Taking Laplace transform of both sides of (1) we have

$L\{D^2y\} + a^2 L(y) = 0$ or, $[p^2L(y) - py(0) - y'(0)] + a^2L(y) = 0$

or, $(p^2 + a^2) L(y) = c_1p + c_2$ where $y(0) = c_1$ and $y'(0) = c_2$

or, $L\{y\} = \dfrac{c_1p + c_2}{p^2 + a^2} = c_1 \cdot \dfrac{p}{p^2 + a^2} + c_2 \cdot \dfrac{1}{p^2 + a^2}$ ... (2)

Now taking inverse Laplace transform of both sides of (2), we have

$$y(t) = c_1 L^{-1}\left\{\frac{p}{p^2 + a^2}\right\} + c_2 L^{-1}\left\{\frac{1}{p^2 + a^2}\right\} = c_1 \cos at + \frac{c_2}{a} \sin at$$

$$= c_1 \cos at + c_3 \sin at, \text{ where } c_3 = \frac{c_2}{a}$$

Hence the required general solution is

$y(t) = c_1\cos at + c_3\sin at$, where c_1 and c_3 are two arbitrary constants.

To verify :

Now differentiating (3) twice with respect to 't'

we get $y'' = -a^2c_1 \cos at - a^2c_3 \sin at$

Now, $D^2y + a^2y = y'' + a^2y = -a^2c_1\cos at - a^2c_3 \sin at + a^2(c_1 \cos at + c_3\sin at) = 0$,

which satisfies (1). So the general solution is $y = c_1\cos at + c_3\sin at$.

25. Solve using Laplace transform :

$$\frac{d^2x}{dt^2} + a^2x = g(t) \text{ given that } x(0) = x'(0) = 0 \text{ and } a > 0.$$

Solution Here the given differential equation is

$$\frac{d^2x}{dt^2} + a^2x = g(t) \quad \text{... (1)}$$

Taking Laplace transform of (1) we have,

$$L\left\{\frac{d^2x}{dt^2}\right\} + a^2L\{x\} = L\{g(t)\}$$

or, $p^2L\{x\} - px(0) - x'(0) + a^2L\{x\} = G(p)$ [where $G(p) = L\{g(t)\}$]

or, $(a^2 + p^2)L\{x\} - p \cdot 0 - 0 = G(p)$ [using $x(0) = x'(0) = 0$]

or, $L\{x\} = \dfrac{G(p)}{a^2 + p^2} = \dfrac{1}{a} \cdot \dfrac{a}{a^2 + p^2} \cdot G(p)$

i.e., $L\{x\} = \dfrac{1}{a} L\{\sin at\} L\{g(t)\}$

Taking inverse Laplace transform, we have,

$x(t) = \dfrac{1}{a} L^{-1}\{L\{\sin at\} L\{g(t)\}\} = \dfrac{1}{a} \int_0^t \sin a(t - x) g(x)\,dx$ [using the convolution theorem]

which is the required solution of the given differential equation.

26. **Solve the following differential equation by Laplace transform :**

$(D^2 - 2aD + a^2)x(t) = f(t)$ where $D \equiv \frac{d}{dt}$.

Solution Here the given differential equation is :

$$(D^2 - 2aD + a^2)x = f(t) \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1) we have,

$$L\{D^2x\} - 2aL\{Dx\} + a^2L\{x\} = L\{f(t)\}$$

or, $p^2L\{x\} - px(0) - x'(0) - 2a[pL\{x\} - x(0)] + a^2L\{x\} = F(p)$ [where $F(p) = L\{f(t)\}$]

or, $(p^2 - 2a + a^2)\bar{x} - pC_0 - C_1 + 2aC_0 = F(p)$

[where $\bar{x} = L\{x\}$ and $x(0) = C_0$ say; $x'(0) = C_1$, say]

or, $(p - a)^2\bar{x} = F(p) + pC_0 + C_1 - 2aC_0$

or, $\bar{x} = \frac{pC_0}{(p-a)^2} + \frac{C_1 - 2aC_0}{(p-a)^2} + \frac{F(p)}{(p-a)^2} = \frac{C_0}{p-a} + \frac{C_1 - aC_0}{(p-a)^2} + \frac{F(p)}{(p-a)^2}$

Taking inverse Laplace transform from both sides, we have,

$$x(t) = C_0L^{-1}\left\{\frac{1}{p-a}\right\} + (C_1 - aC_0)L^{-1}\left\{\frac{1}{(p-a)^2}\right\} + L^{-1}\left\{\frac{F(p)}{(p-a)^2}\right\}$$

$= C_0e^{at} + (C_1 - aC_0)te^{at} + L^{-1}\{L\{f(t)\}L\{te^{at}\}\}$

$= C_0e^{at} + (C_1 - aC_0)te^{at} + \int_0^t (t - x)e^{a(t-x)}f(x)\,dx$ [using the convolution theorem]

$= C_0e^{at} + C_2te^{at} + e^{at}\int_0^t (t - x)e^{-ax}f(x)\,dx$, where $C_2 = C_1 - aC_0$

i.e., $x(t) = e^{at}[C_0 + C_2t + \int_0^t (t - x)e^{-ax}f(x)\,dx]$

which is the required general solution with C_0 and C_2 are two arbitrary constants.

27. **Solve by Laplace transformation :**

$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0,\ y(0) = 1,\ y\left(\frac{\pi}{4}\right) = \sqrt{2}$. **[CP 2020]**

Solution Here the given differential equation is

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0 \quad \text{... (1)}$$

Taking Laplace transform of both sides of (1), we have

$$L\left\{\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y\right\} = L\{0\}$$

or, $L\left\{\frac{d^2y}{dx^2}\right\} + 6L\left\{\frac{dy}{dx}\right\} + 10L\{y\} = 0$ [using linearity property]

or, $\{p^2L\{y\} - py(0) - y'(0)\} + 6\{pL\{y\} - y(0)\} + 10L\{y\} = 0$

[since $L\{f^{(n)}(t)\} = p^2L\{f(t)\} - p^{n-1}f(0) - p^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$]

or, $(p^2 + 6p + 10)L\{y\} - p\cdot 1 - a - 6\cdot 1 = 0$ [since $y(0) = 1$, given and $y'(0) = a$, say]

or, $L\{y\} = \frac{p + a + 6}{p^2 + 6p + 10} = \frac{(p + 3) + (a + 3)}{(p + 3)^2 + 1}$

Hence $y(x) = L^{-1}\left\{\dfrac{(p+3)+(a+3)}{(p+3)^2+1}\right\} = e^{-3x}L^{-1}\left\{\dfrac{p+a+3}{p^2+1}\right\}$ [By first shifting property]

$$= e^{-3x}\left[L^{-1}\left\{\frac{p}{p^2+1}\right\}+(a+3)L^{-1}\left\{\frac{1}{p^2+1}\right\}\right] = e^{-3x}[\cos x + (a+3)\sin x]$$

To determine a, we have from above,

$$y\left(\frac{\pi}{4}\right) = e^{-\frac{3\pi}{4}}\left[\cos\frac{\pi}{4}+(a+3)\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2} \quad \text{[Using the boundary condition]}$$

or, $a + 4 = 2e^{\frac{3\pi}{4}}$ giving $a = 2e^{\frac{3\pi}{4}} - 4$

Hence the required solution is : $y(x) = e^{-3x}[\cos x + (2e^{\frac{3\pi}{4}} - 1)\sin x]$

28. **Find $L^{-1}\left\{\dfrac{4}{s+1}+\dfrac{6}{(s+1)^2}+\dfrac{3}{(s+1)^4}\right\}$ and hence solve $(D^2 + 2D + 1)y = 3te^{-t}$, given $y = 4$, $Dy = 2$ when $t = 0$ and $D \equiv \dfrac{d}{dt}$.** [CP 2020]

Solution Using the first shifting property, we have,

$$L^{-1}\left\{\frac{4}{s+1}+\frac{6}{(s+1)^2}+\frac{3}{(s+1)^4}\right\} = e^{-t}L^{-1}\left\{\frac{4}{s}+\frac{6}{s^2}+\frac{3}{s^4}\right\} = e^{-t}\left[4L^{-1}\left(\frac{1}{s}\right)+6L^{-1}\left\{\frac{1}{s^2}\right\}+3L^{-1}\left\{\frac{1}{s^4}\right\}\right]$$

$$= e^{-t}\left[4\cdot 1+6t+3\cdot\frac{t^3}{6}\right] \quad \left[\text{since } L^{-1}\left\{\frac{n!}{p^{n+1}}\right\} = t^n\right]$$

$$= \frac{e^{-t}}{2}[t^3 + 12t + 8]$$

Therefore $L^{-1}\left\{\dfrac{4}{s+1}+\dfrac{6}{(s+1)^2}+\dfrac{3}{(s+1)^4}\right\} = \dfrac{e^{-t}}{2}[t^3 + 12t + 8]$ ... (1)

Now, the given differential equation is :

$(D^2 + 2D + 1)y = 3te^{-t}$

Taking Laplace transform of both sides of (1) we have,

$$L\{(D^2 + 2D + 1)y\} = L\{3te^{-t}\} \text{ or, } L\{D^2y\} + 2L\{Dy\} + L\{y\} = -3\frac{d}{ds}\left(\frac{1}{s+1}\right)$$

or, $\{s^2L\{y\} - sy(0) - Dy(0)\} + 2\{sL\{y\} - y(0)\} + L\{y\} = \dfrac{3}{(s+1)^2}$

or, $(s^2 + 2s + 1)L\{y\} - 4s - 2 - 8 = \dfrac{3}{(s+1)^2}$ [Using initial conditions]

or, $(s + 1)^2L\{y\} = 4(s + 1) + 6 + \dfrac{3}{(s+1)^2}$

or, $L\{y\} = \dfrac{4}{s+1}+\dfrac{6}{(s+1)^2}+\dfrac{3}{(s+1)^4}$

Hence $y(t) = L^{-1}\left\{\dfrac{4}{s+1}+\dfrac{6}{(s+1)^2}+\dfrac{3}{(s+1)^4}\right\} = \dfrac{e^{-t}}{2}[t^3 + 12t + 8]$ [using (1)]

which is the required solution of the given differential equation.